MATH 2230 A

Midterm I Solution

1a Compute $\frac{3+2i}{2-3i}$ in standard form

Ans: $\frac{3+2i}{2-3i} = \frac{3+2i}{2-3i} \cdot \frac{2+3i}{2+3i}$

$= \frac{13i}{13} = i$

1b Compute $\log(-\sqrt{3}+i)$ in standard form by using principal branch.

Ans: $-\sqrt{3}+i = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 2e^{\frac{5\pi}{6}i}$

$\therefore \log(-\sqrt{3}+i) = \log 2 + 5\pi/6\, i$

1c Compute i^{-i} in standard form by using principal branch.

Ans: $\log i = \pi/2\, i$

$\Rightarrow i^{-i} = e^{-i \log i} = e^{\pi/2}$

2 (1) Show that $f = \begin{cases} \overline{z}^2 & z \neq 0 \\ 0 & z = 0 \end{cases}$ is diff. at and only at $z=0$.

Ans: Suppose $z \in \mathbb{C}$ and $z \neq 0$,

$f(z) = (x-iy)^2 = x^2 - y^2 - 2xyi = u + iv$

$u_x = 2x$, $v_y = -2x$

$u_y = -2y$, $v_x = -2y$ $\Rightarrow u_x, u_y, v_x, v_y \in C$

Thus the Cauchy-Riemann equation can not be satisfied for $z \neq 0$. f is only diff at $z=0$.

(2) (ii) Show that $\sin z$ is diff on $\mathbb{C}$ and find $\frac{d}{dz}\sin z$

Ans: let $\sin z = u + iv$ and by (2) (i)

$$u_x = \cos x \cosh y \qquad v_y = \cos x \cosh y$$

$$u_y = \sin x \sinh y \qquad v_x = -\sin x \sinh y$$

The Cauchy-Riemann equation is satisfied $\forall z \in \mathbb{C}$.

~~Thus it is diff on $\mathbb{C}$ (u, v are C^1).~~

and u, v are C^1, thus it is diff on $\mathbb{C}$.

Since
$$\begin{aligned} f'(z) &= u_x + i v_x \\ &= \cos x \cosh y - i \sin x \sinh y \\ &= \frac{\cos x\,(e^{y}+e^{-y})}{2} - \frac{i \sin x\,(e^{y}-e^{-y})}{2} \\ &= \frac{1}{2}\left(e^{y-ix} + e^{-y+ix}\right) \\ &= \cos z \end{aligned}$$

(3) $\int_C f(z)\,dz$

$$= \int_{-\pi}^{\pi} e^{(a-1)\operatorname{Log}(Re^{i\theta})}\, iRe^{i\theta}\, d\theta$$

$$= iR \int_{-\pi}^{\pi} e^{(a-1)\log R}\, e^{ai\theta}\, d\theta$$

$$= iR^a \int_{-\pi}^{\pi} e^{ai\theta}\, d\theta$$

$$= \frac{iR^a}{ia}\left(e^{ai\pi} - e^{-ai\pi}\right)$$

$$= \frac{2R^a}{a} \sin a\pi$$